Analysis & Linear Algebra

30 subtopics · roughly 60 marks · 35 counterexamples

WHAT THE EXAM ASKS OF EACH SUBTOPIC

The Real Line

Elements of set theory: operations, De Morgan and difference. A free minute at the start of Part B if you rewrite every difference as $A \cap B^c$, and a two-minute case analysis if you do not. The recurring question is $A \setminus (B \setminus C)$, and the recurring wrong answer moves the brackets.

Completeness, sup/inf, Archimedean property. Every question here is really 'which property of $\mathbb{R}$ fails in $\mathbb{Q}$'. Completeness (sup exists) is what separates them; Archimedes and density follow from it.

Sequences: convergence, monotone, Bolzano–Weierstrass, Cauchy. Bolzano–Weierstrass and the Cauchy criterion are tested as *decisions*: given a sequence, is it bounded / Cauchy / convergent / does it have a convergent subsequence? Know exactly which implications hold in $\mathbb{R}$ and which need completeness.

limsup, liminf and subsequential limits. Compute limsup/liminf of explicit sequences fast, and know the algebra: $\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$, with equality if one converges. Convergence $\Leftrightarrow$ limsup = liminf (finite).

Series: comparison, ratio, root, Raabe, condensation, alternating, rearrangements. Know which test to reach for in 10 seconds, the limsup forms, and the three classic traps: conditional vs absolute convergence, rearrangements (Riemann), and 'terms $\to 0$ does not imply convergence'.

Continuity and Differentiation

Continuity, uniform continuity, Lipschitz. The chain Lipschitz $\Rightarrow$ uniformly continuous $\Rightarrow$ continuous, where each converse fails, and what compactness or boundedness of the domain changes. Know $\sqrt{x}$, $1/x$, x^2, $\sin(1/x)$, $\sin(x^2)$ by heart.

Differentiability, mean value theorems, Taylor, L'Hôpital. MVT-family questions are about *which hypothesis is missing*: Rolle needs continuity at the endpoints, Darboux needs only differentiability, L'Hôpital needs the derivative-quotient limit to exist. Know $x^2 \sin(1/x)$ cold.

Integration

Riemann integration and criteria. Lebesgue's criterion (bounded + discontinuities of measure zero) decides Riemann integrability instantly. Know Dirichlet, Thomae, and that composition can destroy integrability.

Improper integrals and convergence tests. Split at every singular point and at infinity, compare with $1/x^p$, and distinguish convergence from absolute convergence — $\int \sin x/x$ is the standard example of the gap.

Sequences and Series of Functions

Pointwise vs uniform convergence, M-test, Dini. Uniform convergence is what lets you swap limit with integral/derivative/continuity. Compute $\sup|f_n - f|$ explicitly; x^n, $nx(1-x)^n$, nxe^{-nx^2}, x/n are the recurring families.

Power series, radius of convergence, Abel's theorem. Radius via limsup $|a_n|^{1/n}$ (Cauchy–Hadamard), never assume the ratio limit exists. Behaviour on the boundary circle is decided separately (Abel); differentiation/integration keep the radius.

Arzelà–Ascoli and equicontinuity. Arzelà–Ascoli = uniformly bounded + equicontinuous ⇒ a uniformly convergent subsequence. It is the compactness criterion in C[a,b]; the failures are x^n (not equicontinuous) and constants n (not bounded).

Functions of Several Variables

Partial derivatives, differentiability, chain rule. Partials existing ⇏ continuous ⇏ differentiable. The safe implication is: continuous partials ⇒ differentiable. Know $xy/(x^2+y^2)$ and the equality-of-mixed-partials failure.

Inverse and implicit function theorems, extrema. Both theorems need a non-vanishing Jacobian ($\partial F/\partial y \neq 0$ for implicit). Where it vanishes, anything can happen — that is exactly what the questions probe.

Metric Spaces

Open/closed sets, limit points, closure, interior. Sets can be both open and closed, or neither. Know which operations preserve openness (arbitrary unions, finite intersections) and the standard $\mathbb{Q}$ examples.

Compactness: open covers, sequential, Heine–Borel. Closed + bounded ⇒ compact ONLY in $\mathbb{R}^n$. Know the ℓ^2 unit ball and discrete $\mathbb{R}$ as spoilers. Compact ⇒ complete ⇒ closed.

Completeness and Baire category. Completeness: closed subsets of complete spaces, ℓ^p, C[0,1] with sup norm are complete; $\mathbb{Q}$, (0,1), C[0,1] with L^1 norm are not. Baire: $\mathbb{R}$ is not a countable union of nowhere dense sets; a complete metric space without isolated points is uncountable.

Connectedness and path-connectedness. Connected subsets of $\mathbb{R}$ are intervals; continuous images stay connected; path-connected ⇒ connected with the topologist's sine curve as the standard converse failure — except open subsets of $\mathbb{R}^n$, where the two agree.

Lebesgue Measure and Integration

Measurable sets and functions. Measure zero, countable additivity and 'almost everywhere' are the workhorses. Know that measurable ⊋ Borel and that a non-measurable set requires the axiom of choice.

Lebesgue integral, MCT, DCT, Fatou. Pick the right convergence theorem: MCT (increasing, non-negative), Fatou (inequality, always), DCT (needs a dominating integrable g). The moving-bump examples show what happens without domination.

L^p spaces essentials. L^p inclusions go one way on finite measure spaces and the other way for ℓ^p — getting the direction right is most of the battle.

Vector Spaces and Linear Maps

Bases, dimension, rank–nullity. Rank–nullity plus 'rank is unchanged by field extension' answers most of these. Watch the direction of the rank inequalities for products.

Linear transformations, matrix representation, change of basis. Similar matrices are the same operator in different bases: they share rank, trace, determinant, characteristic and minimal polynomials — but sharing those is not enough to be similar.

Eigenvalues and Canonical Forms

Eigenvalues, characteristic & minimal polynomials, Cayley–Hamilton. Read eigen-structure off the characteristic and minimal polynomials without computing: diagonalisable ⇔ minimal polynomial has distinct linear factors; similar matrices share both polynomials but the converse fails; Cayley–Hamilton lets you compute inverses and high powers.

Diagonalisability criteria. Decide diagonalisability from a single equation or property: idempotent, involution, $A^k = I$, nilpotent, symmetric, normal, distinct eigenvalues. Know over which field.

Jordan canonical form. Recover the block structure from three numbers per eigenvalue: algebraic multiplicity (total size), geometric multiplicity (number of blocks), minimal-polynomial exponent (largest block).

Rational canonical form. Rational canonical form works over any field — use it when the characteristic polynomial does not split. Invariant factors divide one another; the last one is the minimal polynomial.

Inner Product Spaces and Forms

Gram–Schmidt, orthogonal/unitary/normal matrices, spectral theorem. Spectral theorem: real symmetric ⇒ orthogonally diagonalisable with real eigenvalues; complex normal ⇒ unitarily diagonalisable. Know which matrix classes are normal.

Quadratic forms, positive definiteness, Sylvester's law. Signature is the complete invariant over $\mathbb{R}$ (Sylvester). Positive definiteness is tested by leading principal minors, or by all eigenvalues being positive — do not mix the two tests up.

Determinants and Matrix Tricks

Determinants, trace, block matrices, rank inequalities. Block formulas and the trace/determinant identities turn hard computations into one-liners. $\det(I + AB) = \det(I + BA)$ is the single most useful trick here.

TEMPTING, AND FALSE

Each claim below feels true and is not. The object beside it is the one that settles it.

> **Cesàro means converge ⇒ the sequence converges**
> ↳ $a_n = (-1)^n$ — Partial averages → 0, sequence diverges.

> **$\limsup(a_n + b_n) = \limsup a_n + \limsup b_n$**
> ↳ $a_n = (-1)^n$, $b_n = (-1)^{n+1}$ — $a_n + b_n \equiv 0$, so the left side is 0 while the right side is $1 + 1 = 2$.

> **$a_n \to 0 \Rightarrow \Sigma a_n$ converges**
> ↳ Harmonic series $\Sigma 1/n$

> **Σa_n converges ⇒ Σa_n^2 converges**
> ↳ $a_n = (-1)^n/\sqrt{n}$ — Alternating series converges; squares give the harmonic series.

> **Continuous on a bounded interval ⇒ bounded**
> ↳ $f(x) = 1/x$ on $(0,1)$ — Needs a compact (closed) domain.

> **Bounded and continuous on $\mathbb{R}$ ⇒ uniformly continuous**
> ↳ $f(x) = \sin(x^2)$ — $x_n = \sqrt{2\pi n}$, $y_n = \sqrt{2\pi n + \pi/2}$ satisfy $|x_n - y_n| \to 0$ while $|f(x_n) - f(y_n)| = 1$.

> **If lim f/g exists (0/0 form) then lim f′/g′ exists**
> ↳ $f(x) = x^2 \sin(1/x)$, $g(x) = x$ as $x \to 0$ — $f/g = x \sin(1/x) \to 0$, but $f'/g' = 2x \sin(1/x) - \cos(1/x)$ has no limit. L'Hôpital goes one way only.

> **$|f|$ Riemann integrable ⇒ f Riemann integrable**
> ↳ $f = 1$ on $\mathbb{Q} \cap [0,1]$, -1 elsewhere — $|f| \equiv 1$ is integrable; f is discontinuous everywhere.

> **If $\int_0^\infty f$ converges then $f(x) \to 0$**
> ↳ f with a spike of height n and width $2/n^3$ at each integer n — The total area is finite but f is unbounded, so it does not tend to 0.

$f_n \to f$ uniformly $\Rightarrow f_n' \to f'$

↳ $f_n(x) = \sin(nx)/n$ — Converges uniformly to 0, derivatives $\cos(nx)$ do not converge.

$\Sigma a_n x^n \to L$ as $x \to 1^-$ $\Rightarrow \Sigma a_n = L$

↳ $\Sigma(-1)^n x^n = 1/(1+x) \to 1/2$ — $\Sigma(-1)^n$ diverges. Abel's theorem has no converse without a Tauberian condition.

A uniformly bounded sequence in C[0,1] has a uniformly convergent subsequence

↳ $f_n(x) = x^n$ — Bounded by 1, but the pointwise limit is discontinuous so no subsequence converges uniformly. Equicontinuity is the missing hypothesis.

If all partial derivatives exist at a point then f is continuous there

↳ $f(x,y) = xy/(x^2+y^2)$, $f(0,0) = 0$ — Both partials are 0 at the origin, but f = ½ along y = x, so f is not continuous.

A C^1 map with everywhere non-zero Jacobian is injective

↳ $f(x,y) = (e^x \cos y, e^x \sin y)$ on $\mathbb{R}^2$ — The Jacobian determinant is $e^{2x} \neq 0$, but $f(x, y) = f(x, y + 2\pi)$. Invertibility is only local.

An arbitrary intersection of open sets is open

↳ $\bigcap_{n\geq 1}(-1/n, 1/n) = \{0\}$ — Only finite intersections preserve openness.

Closed and bounded ⇒ compact

↳ Closed unit ball in ℓ^2 (or in C[0,1] with sup norm) — $e_1, e_2, \ldots$ has no convergent subsequence since $\|e_n - e_m\| = \sqrt{2}$. Heine–Borel is $\mathbb{R}^n$-only.

Bounded ⇒ totally bounded

↳ $\mathbb{R}$ with the discrete metric — Everything is within distance 1, but no finite set of balls of radius ½ covers it.

Completeness is a topological property

↳ (0,1) and $\mathbb{R}$ — Homeomorphic, but $\mathbb{R}$ is complete and (0,1) is not (1/n is Cauchy).

$d(Tx, Ty) < d(x, y)$ for all $x \neq y$ on a complete space ⇒ T has a fixed point

↳ $T(x) = x + 1/x$ on $[1, \infty)$ — Distances strictly decrease but no contraction constant $k < 1$ exists; no fixed point.

Connected ⇒ path-connected

↳ Topologist's sine curve $\{(x, \sin 1/x) : 0 < x \leq 1\} \cup \{0\}\times[-1,1]$ — Connected as the closure of a connected set; no path reaches the segment.

Closure of a path-connected set is path-connected

↳ Graph of $\sin(1/x)$ on (0,1] — Its closure is the topologist's sine curve.

A set of measure zero is countable

↳ The Cantor set — Uncountable, yet measure zero.

Pointwise convergence implies convergence of the integrals

↳ $f_n = n\cdot 1_{(0,1/n)}$ on [0,1] — $f_n \to 0$ pointwise but $\int f_n = 1$ always. Domination or monotonicity is essential.

$L^1[0,1] \subseteq L^2[0,1]$

↳ $f(x) = 1/\sqrt{x}$ — $\int f = 2 < \infty$ but $\int f^2 = \int dx/x = \infty$. On a finite measure space the inclusion runs the other way.

An injective linear operator on a vector space is surjective

↳ The right shift on ℓ^2: $(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$ — Injective but misses everything with a non-zero first coordinate. Rank–nullity needs finite dimension.

Equivalent matrices are similar
↳ $A = I_2$ and $B = 2I_2$ — Both are invertible, so each is P·(the other)·Q for suitable invertible P, Q — every invertible n×n matrix is equivalent to every other, because equivalence sees nothing finer than rank. But A has eigenvalue 1 and B has eigenvalue 2, and similarity preserves eigenvalues. Equivalence allows an independent change of basis in the domain and the codomain; similarity forces the same basis on both sides.

Same characteristic polynomial ⇒ similar
↳ 0 matrix and [[0,1],[0,0]] — Both have char poly x^2, different minimal polynomials (x vs x^2).

Same characteristic and minimal polynomial ⇒ similar
↳ 4×4 nilpotent matrices with Jordan blocks of sizes {2, 2} and {2, 1, 1} — Both have $\chi = x^4$ and $m = x^2$, but different numbers of blocks (ranks 2 vs 1).

AB and BA have the same minimal polynomial
↳ A = [[0,1],[0,0]], B = [[0,0],[0,1]] — AB = A has minimal polynomial x^2, BA = 0 has x. The characteristic polynomials do agree.

Real matrix with real eigenvalues is diagonalisable
↳ [[1,1],[0,1]] — Eigenvalue 1 with geometric multiplicity 1 < algebraic 2.

Commuting matrices are simultaneously diagonalisable
↳ A = B = [[0,1],[0,0]] — They commute but neither is diagonalisable. Need each to be diagonalisable first.

Every real matrix has a Jordan form over $\mathbb{R}$
↳ The rotation [[0,−1],[1,0]] — Its eigenvalues $\pm i$ are not real; over $\mathbb{R}$ one uses the real Jordan form with a 2×2 rotation block.

A real matrix with all real eigenvalues is orthogonally diagonalisable
↳ [[1,1],[0,1]] — Eigenvalue 1 twice but only one eigenvector; orthogonal diagonalisability requires symmetry.

det A > 0 implies A is positive definite
↳ A = diag(−1, −1) — det = 1 > 0 but both eigenvalues are negative. All leading principal minors must be positive.

There exist matrices with AB − BA = I
↳ Impossible over $\mathbb{R}$ or $\mathbb{C}$ in finite dimensions — trace(AB − BA) = 0 but trace(I) = $n \neq 0$. (It is possible for unbounded operators — the Heisenberg relation.)
