Complex Analysis, Algebra & Topology

24 subtopics · roughly 50 marks · 26 counterexamples

WHAT THE EXAM ASKS OF EACH SUBTOPIC

Analytic Functions

Cauchy–Riemann equations, harmonic functions. CR equations alone do not give holomorphy — you need them plus continuity of the partials (or real-differentiability). The standard trap is a function satisfying CR only at the origin.

Power series and analyticity. Holomorphic ⇔ analytic ⇔ locally a convergent power series — an equivalence with no real-analysis analogue. Use the identity theorem to force $f \equiv g$ from agreement on a set with a limit point *inside* the domain.

Cauchy Theory

Cauchy's theorem and integral formula. Pick the right tool: Cauchy's theorem (no singularities inside) vs the integral formula (one simple pole) vs residues (several). Simple connectedness is what makes $\oint = 0$.

Liouville, Morera, maximum modulus principle. Recognise Liouville in disguise: bounded, bounded real part, bounded on a growth scale, or missing two values. Maximum modulus turns interior bounds into boundary bounds.

Singularities and Residues

Laurent series, classification of singularities, Casorati–Weierstrass. Classify a singularity from the Laurent tail or from the behaviour of $|f|$ nearby: bounded ⇒ removable, → ∞ ⇒ pole, neither ⇒ essential (and then the image of every punctured neighbourhood is dense).

Residue theorem and standard contour integrals. Five standard contour patterns cover almost every exam integral. Know which contour goes with which integrand, and remember that a semicircular indentation contributes $i\pi\cdot\text{Res}$, not $2\pi i\cdot\text{Res}$.

Zeros and Mappings

Argument principle, Rouché's theorem, open mapping. Rouché is a counting tool: split the polynomial into a dominant term and the rest, verify the strict inequality on the circle, and read off the zero count. Watch for the extra factor when the integrand is $(zf)'/(zf)$.

Conformal maps, Möbius transformations, Schwarz lemma. Know the dictionary of standard maps between disc, half-plane, strip and sector, and that Möbius maps send circles-and-lines to circles-and-lines and preserve cross-ratio.

Groups

Subgroups, cosets, Lagrange, cyclic groups. Lagrange gives one direction only. The exam tests the failures: no subgroup of a given order (A_4), and 'union of proper subgroups ⇔ not cyclic'.

Normal subgroups, quotients, isomorphism theorems. Normality is not transitive. Know the standard normal/non-normal examples in S_4 and A_4, and that index-2 subgroups are always normal.

Permutation groups: cycles, sign, conjugacy in S_n and A_n. Cycle type determines conjugacy in S_n (but splits in A_n). Count elements of a given order by counting cycle types; know the order of a permutation is the lcm of its cycle lengths.

Group actions, class equation, p-groups. The class equation is the engine: it proves p-groups have non-trivial centre, and drives the 'no simple group of order n' arguments.

Sylow theorems and groups of small order. $n_p \equiv 1 \pmod p$ and $n_p \mid m$ is the whole toolkit. Use it to force a normal Sylow subgroup and prove non-simplicity, or to classify groups of small order.

Finite abelian groups. Convert between invariant-factor and elementary-divisor form fast, and count subgroups/elements of a given order using the partition structure.

Rings and Fields

Ideals, quotient rings, prime & maximal ideals, CRT. Prime ⇔ integral domain quotient, maximal ⇔ field quotient. CRT converts counting questions mod n into products over prime powers.

Euclidean, PID, UFD hierarchy. Memorise the chain and the counterexample at each strict inclusion — that single table answers most Part-C ring questions.

Polynomial rings and irreducibility tests. Pick the right test: rational root, Eisenstein (possibly after a shift), or reduction mod p. Irreducible over $\mathbb{Q}$ does not mean irreducible mod every p.

Field extensions, splitting fields, finite fields. Tower law plus 'degree = degree of the minimal polynomial' answers most questions. Know the standard degrees: $[\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}] = 6$, $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.

Galois theory essentials. For an exam you need the Galois groups of $x^3 - 2$, $x^4 - 2$, $x^n - 1$ and the fundamental correspondence — subgroups ↔ intermediate fields, with normal subgroups ↔ normal extensions.

Topology

Topological spaces, bases, subspace/product/quotient. Compare topologies (finer/coarser) and know how the product topology differs from the box topology on infinite products — that difference is examined directly.

Continuity, homeomorphism, separation axioms. Know the implication chain $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$ (with the regularity/normality convention) and which properties are hereditary or productive.

Compactness and Tychonoff. Heine–Borel is $\mathbb{R}^n$-only. In general spaces use open covers or the finite intersection property; in metric spaces sequential compactness is equivalent.

Connectedness and components. Components are always closed but not always open; in $\mathbb{Q}$ every component is a single point. Use clopen sets or a continuous surjection onto {0,1} to disprove connectedness.

Standard spaces: cofinite, cocountable, Sorgenfrey, Cantor set. These four spaces are the exam's stock counterexamples — memorise the property table and you can answer most 'which of the following is true' topology items by elimination.

TEMPTING, AND FALSE

Each claim below feels true and is not. The object beside it is the one that settles it.

> **The Cauchy–Riemann equations at a point imply complex differentiability there**
> ↳ $f(z) = (\bar{z})^2/z$ for $z \neq 0$, $f(0) = 0$ — CR hold at 0, but the difference quotient along $z = t(1+i)$ differs from the one along the real axis.

> **Zeros of a non-constant holomorphic function cannot accumulate**
> ↳ $\sin(1/z)$ on $\mathbb{C}\setminus\{0\}$ — Zeros $1/(n\pi)$ accumulate at 0, which lies outside the domain. Inside the domain zeros are always isolated.

Cauchy's theorem applies to any domain without singularities of f

↳ $f(z) = 1/z$ on the annulus $½ < |z| < 2$ — f is holomorphic there but $\oint_{|z|=1} dz/z = 2\pi i \neq 0$ — the annulus is not simply connected.

A bounded holomorphic function on an unbounded domain is constant

↳ $f(z) = e^z$ on $\{\mathrm{Re}\, z < 0\}$ — $|e^z| < 1$ there, but f is not constant. Liouville needs the whole plane.

|f| bounded near an isolated singularity ⇒ pole

↳ $f(z) = \sin(z)/z$ at 0 — Bounded ⇒ removable (Riemann). Poles have $|f| \to \infty$.

If |f| is bounded near an isolated singularity, the singularity is a pole

↳ $f(z) = \sin z / z$ at 0 — Bounded ⇒ removable (Riemann). Poles have $|f| \to \infty$.

A small indentation around a simple pole contributes 2πi·Res

↳ The indentation at 0 when computing $\int_0^\infty \sin x/x\, dx$ — A half-circle contributes iπ·Res; using 2πi gives twice the right answer.

$f'(z) \neq 0$ everywhere implies f is injective

↳ $f(z) = e^z$ on $\mathbb{C}$ — $f' = e^z$ never vanishes, yet $f(z) = f(z + 2\pi i)$. Non-vanishing derivative gives only local injectivity.

$\mathbb{C}$ and the unit disc are biholomorphic (both are simply connected)

↳ Liouville's theorem — A biholomorphism $\mathbb{D} \to \mathbb{C}$ would invert to a bounded entire function. $\mathbb{C}$ is the sole exception in the Riemann mapping theorem.

If d divides |G| then G has a subgroup of order d

↳ A_4 has order 12 but no subgroup of order 6

Converse of Lagrange: d | |G| ⇒ subgroup of order d

↳ A_4 has no subgroup of order 6

$H \trianglelefteq K$ and $K \trianglelefteq G$ imply $H \trianglelefteq G$

↳ $\langle(12)(34)\rangle \trianglelefteq V_4 \trianglelefteq A_4$ — Conjugating (12)(34) by the 3-cycle (123) gives $(13)(24) \notin \langle(12)(34)\rangle$.

A_n is simple for every $n \geq 3$

↳ A_4 has the normal subgroup $V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$

G/Z(G) can be cyclic and non-trivial

↳ Impossible — if G/Z is cyclic then G is abelian, so G/Z is trivial — This 'no counterexample exists' fact is itself examined: |G/Z| is never prime.

A group of order pq (p < q) is always cyclic

↳ S_3 of order 6 = 2·3 — Non-abelian because 2 | 3 − 1. The cyclic conclusion needs $p \nmid q - 1$.

Every prime ideal is maximal

↳ (X) in $\mathbb{Z}[X]$, or (0) in any integral domain that is not a field — $\mathbb{Z}[X]/(X) \cong \mathbb{Z}$ is a domain but not a field. In a PID the implication does hold for non-zero primes.

Every integral domain is a UFD

↳ $\mathbb{Z}[\sqrt{-5}]$: $6 = 2\cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$

In every integral domain, irreducible implies prime

↳ 2 in $\mathbb{Z}[\sqrt{-5}]$ — 2 is irreducible (no element of norm 2) but divides $(1+\sqrt{-5})(1-\sqrt{-5}) = 6$ without dividing either factor.

A polynomial irreducible over $\mathbb{Q}$ is irreducible modulo some prime

↳ $x^4 + 1$ — Irreducible over $\mathbb{Q}$, yet reducible modulo every prime.

An algebraic extension is a finite extension
↳ The field of all algebraic numbers $\bar{\mathbb{Q}}$ over $\mathbb{Q}$ — Every element is algebraic, but the extension has infinite degree (it contains $\mathbb{Q}(2^{1/n})$ for every n).

Every extension of degree n has a Galois group of order n
↳ $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ has degree 3 but only the identity automorphism — The extension is not normal — the other cube roots are not real. Order = degree only for Galois extensions.

$\mathrm{cl}(A \cap B) = \mathrm{cl}(A) \cap \mathrm{cl}(B)$
↳ $A = \mathbb{Q}$, $B = \mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$ — $\mathrm{cl}(A \cap B) = \mathrm{cl}(\emptyset) = \emptyset$ but $\mathrm{cl}\,A \cap \mathrm{cl}\,B = \mathbb{R}$.

A continuous bijection is a homeomorphism
↳ $\mathrm{id} : (\mathbb{R}, \text{discrete}) \to (\mathbb{R}, \text{usual})$, or $t \mapsto (\cos t, \sin t)$ from $[0, 2\pi)$ to S^1 — Needs compact domain and Hausdorff codomain.

The closed unit ball of a normed space is compact
↳ The unit ball of ℓ^2 — Riesz: compactness of the ball holds exactly in finite dimensions.

Connected components are open
↳ $\mathbb{Q}$ with the usual topology — Components are singletons, which are not open. They are open exactly when the space is locally connected.

A compact T_1 space is Hausdorff
↳ An infinite set with the cofinite topology — Compact and T_1 but any two non-empty open sets intersect.

NETMaths · netmaths.in/cheat-sheet/complex-analysis-algebra-topology — free to print and share. The full chapters, solved previous-year questions and mock tests are on the site.