ODE, PDE & Applied Mathematics

16 subtopics · roughly 45 marks · 12 counterexamples

WHAT THE EXAM ASKS OF EACH SUBTOPIC

Ordinary Differential Equations

Existence–uniqueness, Picard, Lipschitz. Peano gives existence from continuity alone; uniqueness needs a Lipschitz condition in y. Global existence needs the right side to grow at most linearly. $y' = y^{1/3}$ and $y' = y^2$ are the two spoilers.

Linear ODE, Wronskian, variation of parameters, systems. Wronskian non-zero ⇒ independent, but the converse needs the functions to solve a common linear ODE. For systems, the eigenvalues of A decide the growth rate of e^{At}.

Sturm–Liouville problems and Green's functions. Eigenvalues of a regular Sturm–Liouville problem are real, simple and unbounded above; eigenfunctions for distinct eigenvalues are orthogonal with respect to the weight. Green's function is built from the two boundary solutions.

Stability and phase portraits. Classify a 2×2 linear system from trace and determinant alone; for nonlinear systems linearise and use Hartman–Grobman — but remember the centre case is the one linearisation cannot decide.

Partial Differential Equations

First-order PDE: Lagrange, Charpit, characteristics. Write the characteristic ODEs, carry the initial data along them, and check whether characteristics cross — that crossing is what makes the solution fail to exist globally.

Classification and canonical forms. $B^2 - 4AC$ decides the type pointwise; the sign can change with the point (Tricomi). Reduce to canonical form by solving the characteristic ODE $dy/dx = (B \pm \sqrt{B^2-4AC})/2A$.

Laplace, heat and wave equations: separation of variables. Know d'Alembert cold, the separation-of-variables series for the three classical equations, and the qualitative differences: smoothing (heat), finite speed (wave), mean value (Laplace).

Numerical Analysis

Root finding: bisection, Newton–Raphson, fixed point, order of convergence. Order of convergence is the whole topic: bisection 1, secant $\varphi \approx 1.618$, Newton 2 (but only 1 at a multiple root), fixed point 1 unless $g'(r) = 0$.

Interpolation and numerical integration with error terms. Degree of precision is the key number: trapezoidal 1, Simpson 3, n-point Gauss 2n−1. Interpolation error carries $f^{(n+1)}/(n+1)!$ times the node product.

Numerical ODE: Euler, Runge–Kutta. Know the local vs global order (Euler: local $O(h^2)$, global $O(h)$), the RK2 order conditions, and the stability interval for Euler applied to $y' = \lambda y$.

Calculus of Variations

Euler–Lagrange equation and standard functionals. Write the Euler–Lagrange equation; use the Beltrami identity when F has no explicit x. Terms that are exact derivatives (null Lagrangians) do not change the extremals.

Isoperimetric problems. Introduce a Lagrange multiplier, solve the modified Euler–Lagrange equation, then use the constraint plus boundary conditions to pin every constant.

Linear Integral Equations

Fredholm and Volterra equations. Volterra always has a unique solution; Fredholm depends on whether λ is an eigenvalue, and then the alternative decides solvability by an orthogonality condition.

Separable kernels and resolvent kernels. Separable kernels turn integral equations into matrix problems. The resolvent kernel sums the Neumann series and gives the solution in closed form.

Classical Mechanics

Lagrangian formalism and generalised coordinates. Write T and V in the generalised coordinate, form $L = T - V$, and apply $d/dt(\partial L/\partial \dot{q}) = \partial L/\partial q$. Cyclic coordinates give conserved momenta immediately.

Hamiltonian formalism and conservation laws. $H = \Sigma p\dot{q} - L$ expressed in (q, p). Poisson brackets test canonicity: $\{Q_i, P_j\} = \delta_{ij}$, $\{Q_i, Q_j\} = \{P_i, P_j\} = 0$.

TEMPTING, AND FALSE

Each claim below feels true and is not. The object beside it is the one that settles it.

> Every IVP has a unique solution
> ↳ $y' = y^{1/3}$, $y(0) = 0$ — Not Lipschitz at 0; $y \equiv 0$ and $y = (2x/3)^{3/2}$ both solve it.

> A continuous right-hand side gives a unique solution
> ↳ $y' = y^{1/3}$, $y(0) = 0$ — Not Lipschitz at 0: $y \equiv 0$ and $y = (2x/3)^{3/2}$ both solve it, as do infinitely many hybrids.

> $W(f, g) \equiv 0$ implies f and g are linearly dependent
> ↳ $f(x) = x^2$, $g(x) = x|x|$ on $\mathbb{R}$ — The Wronskian vanishes identically but no constant multiple relates them. The implication holds only for solutions of a common linear ODE.

> Every boundary value problem has a Green's function
> ↳ $y'' + \pi^2 y = f$ on $[0,1]$ with $y(0) = y(1) = 0$ — $\lambda = \pi^2$ is an eigenvalue of the homogeneous problem, so the operator is not invertible and no Green's function exists.

> Linearisation determines stability at every equilibrium
> ↳ $\dot{x} = -y - x^3$, $\dot{y} = x - y^3$ at the origin — The linearisation is a centre (eigenvalues $\pm i$), but $V = x^2 + y^2$ gives $\dot{V} = -2(x^4 + y^4) < 0$: asymptotically stable. Non-hyperbolic equilibria escape Hartman–Grobman.

> A first-order quasilinear Cauchy problem with smooth data has a global smooth solution
> ↳ $u u_x + u_y = 0$ with $u(x, 0) = -x$ — Characteristics $x = x_0(1 - y)$ all meet at $y = 1$: the solution blows up in gradient (a shock).

> A PDE has a single type throughout its domain
> ↳ The Tricomi equation $y u_{xx} + u_{yy} = 0$ — Elliptic in the upper half-plane, hyperbolic in the lower, parabolic on the axis.

> Newton's method converges quadratically to any root
> ↳ $f(x) = x^2$ at the root 0 — The root is double: $x_{n+1} = x_n/2$, only linear convergence.

> Increasing the number of equally spaced interpolation nodes improves the approximation
> ↳ Runge's function $1/(1 + 25x^2)$ on $[-1, 1]$ — The interpolants diverge near the endpoints as $n \to \infty$. Chebyshev nodes restore convergence.

A weak minimum of a functional is a strong minimum

↳ $J[y] = \int_0^\pi (1 - (y')^2)y^2\,dx$ at $y \equiv 0$ — In the C^1-ball the integrand is non-negative so $J \geq 0$; in the C^0-ball a steeply oscillating small y makes J negative.

Every integral equation of the second kind has a unique solution

↳ $y(x) - 3\int_0^1 tx\, y(t)dt = e^x$ — $\lambda = 3$ is an eigenvalue of the kernel tx, and e^x is not orthogonal to t — no solution exists.

The Hamiltonian always equals the total energy

↳ A bead on a wire rotating at forced angular velocity ω — The constraint is time-dependent, so H is conserved but differs from T + V.

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