Probability, Statistics & Linear Programming

15 subtopics · roughly 45 marks · 13 counterexamples

WHAT THE EXAM ASKS OF EACH SUBTOPIC

Probability

Axioms, conditional probability, independence, Bayes. Most Part-C items here are symmetry checks: $P(A|B) > P(A) \Leftrightarrow P(A \cap B) > P(A)P(B)$ is symmetric in A and B, while $P(A|B) > P(B)$ is not.

Random variables, distributions, moments, MGF. MGF determines the distribution and factorises over independent sums. Know which distributions have no MGF (Cauchy, t) and which moments fail to exist.

Standard discrete and continuous distributions. Know the mean/variance table cold and the standard relationships (sum of exponentials = gamma, min of exponentials = exponential, memorylessness).

Joint distributions, transformations, order statistics. Uncorrelated is weaker than independent except for the joint normal. Use the Jacobian formula for transformations and the standard order-statistic densities.

Limit Theorems and Markov Chains

Modes of convergence, WLLN, SLLN, CLT. Know the implication diagram and one counterexample per missing arrow. The typewriter sequence and the moving-bump are the two you need.

Markov chains: classification of states, stationary distributions. Classify states (recurrent/transient, periodicity), then use irreducible + aperiodic + positive recurrent ⇒ unique stationary distribution with $\pi_j = 1/m_{jj}$.

Estimation

Sufficiency, completeness, UMVUE, Cramér–Rao. Factorisation gives sufficiency; Lehmann–Scheffé turns a complete sufficient statistic plus unbiasedness into the UMVUE. Cramér–Rao gives a bound that is often not attained.

MLE and method of moments. MLE is invariant and asymptotically efficient but can be biased and need not be unique; moment estimators are easy but usually inefficient.

Hypothesis Testing

Neyman–Pearson lemma and UMP tests. NP gives the most powerful test for simple-vs-simple; monotone likelihood ratio upgrades it to UMP for one-sided alternatives. Two-sided alternatives usually have no UMP test.

Likelihood ratio and standard tests. Match the situation to the test and remember the degrees of freedom. For confidence intervals, watch whether the quantile is upper or lower and whether σ is known.

Linear Models and Multivariate

Gauss–Markov, regression, ANOVA basics. Gauss–Markov needs uncorrelated errors of equal variance; with heteroscedasticity the BLUE is weighted least squares. Estimability in ANOVA is decided by whether the function is a combination of cell means.

Multivariate normal distribution. Linear combinations of a multivariate normal are normal — that single fact plus the Wishart quadratic-form rule answers most questions here.

Sampling and Design of Experiments

SRS, stratified and systematic sampling. Know when SRSWOR beats SRSWR (the finite population correction), and that proportional allocation beats SRS while Neyman allocation beats proportional.

CRD, RBD, LSD essentials. Degrees of freedom are the most-asked item: CRD $N - k$, RBD $(k-1)(r-1)$, LSD $(p-1)(p-2)$. Blocking removes a nuisance source and costs error degrees of freedom.

Linear Programming

Linear programming, simplex and duality. Part C sets these as multi-statement questions about a single programme: whether it is feasible, whether the optimum is attained, whether it is unbounded, and what the dual says. The arithmetic is light; the marks turn on reading the constraints exactly.

TEMPTING, AND FALSE

Each claim below feels true and is not. The object beside it is the one that settles it.

Pairwise independent events are mutually independent
↳ Two fair coin tosses: A = first is heads, B = second is heads, C = the two agree — Each pair is independent, but $P(A\cap B\cap C) = 1/4 \neq 1/8 = P(A)P(B)P(C)$.

Every random variable has a moment generating function
↳ The standard Cauchy distribution — $E[e^{tX}] = \infty$ for every $t \neq 0$; even $E|X|$ is infinite. Its characteristic function $e^{-|t|}$ exists.

The maximum of independent exponentials is exponential
↳ $\max(X_1, X_2)$ with $X_i \sim \mathrm{Exp}(1)$ — The *minimum* is exponential (rate $\lambda_1+\lambda_2$); the maximum has CDF $(1 - e^{-x})^2$, which is not exponential.

If X and Y are each normal and uncorrelated then they are independent
↳ $X \sim N(0,1)$, $\varepsilon = \pm 1$ with probability ½ independent of X, $Y = \varepsilon X$ — Y is $N(0,1)$, $\mathrm{Cov}(X,Y) = 0$, but $|X| = |Y|$ always. The pair is not jointly normal.

Convergence in probability implies almost sure convergence
↳ The typewriter sequence on $[0,1]$ — $P(X_n \neq 0) \to 0$ but every ω is hit infinitely often, so there is no a.s. limit.

An irreducible chain with a stationary distribution converges to it
↳ The two-state chain that swaps deterministically (period 2) — $\pi = (½, ½)$ is stationary and unique, but p_{nij} oscillates between 0 and 1. Aperiodicity is required.

A sufficient statistic is complete
↳ $X_{(n)}$ (or $(X_{(1)}, X_{(n)})$) for Uniform$(-\theta, \theta)$ — Sufficient but not complete: symmetry gives non-zero functions with zero expectation.

The MLE is unbiased
↳ $\hat{\sigma}^2 = (1/n)\Sigma(X_i - \bar{X})^2$ for $N(\mu, \sigma^2)$, or $X_{(n)}$ for Uniform$(0,\theta)$ — Both underestimate systematically; $E[X_{(n)}] = n\theta/(n+1)$.

A UMP test exists for every testing problem
↳ $H_0: \mu = 0$ vs $H_1: \mu \neq 0$ for $N(\mu, 1)$ — The MP test for $\mu > 0$ rejects for large $\bar{X}$, for $\mu < 0$ for small $\bar{X}$; no single test is best against both.

Any interval of the form $[\bar{X} - (S/\sqrt{n})t, \infty)$ with a 90% quantile t is a 90% confidence interval
↳ $t = t_{n-1,0.9}$ under the convention $t_{m,\alpha} = (1-\alpha)$-th quantile — That t is the 10th percentile, so the interval has coverage 10%, not 90%.

OLS is the BLUE in every linear model

↳ $Y_i = \beta x_i + \varepsilon_i$ with $\mathrm{Var}(\varepsilon_i) = \sigma^2 x_i^2$ — Gauss–Markov assumes constant variance; here weighted least squares (the mean of Y_i/x_i) has smaller variance.

If every marginal is normal then the vector is multivariate normal

↳ $X \sim N(0,1)$ and $Y = \varepsilon X$ with $\varepsilon = \pm 1$ independent — Both marginals are $N(0,1)$ but $X + Y$ is 0 half the time — not normal, so the pair is not jointly normal.

Systematic sampling is always at least as efficient as SRS

↳ A population with a periodic pattern of period k, sampled every k-th unit — Every sampled unit falls at the same phase, so the sample can be maximally unrepresentative.

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